Description of Vorticity by Grassmann Variables and an Extension to Supersymmetry

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Typeset in LaTeX by M. Stock

MIT-CTP#3036

Abstract

In this paper particle physics concepts are blended into a field theory for macroscopic phenomena: Fluid mechanics is enhanced by anticommuting Grassmann variables to describe vorticity, while an additional interaction for the Grassmann variables leads to supersymmetric fluid mechanics.

1 Précis of Fluid Mechanics (With No Vorticity)

Let me begin with a précis of fluid mechanical equations [1]. An isentropic fluid is described by a matter density field ρ and a velocity field $\boldsymbol{v}$, which satisfy a continuity equation involving the current $\boldsymbol{j} = \rho\boldsymbol{v}$:

$$\dot{\rho} + \boldsymbol{\nabla} \cdot (\rho\boldsymbol{v}) = 0 \tag{1}$$

and a force equation involving the pressure P:

$$\dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P \ . \tag{2}$$

(Over-dot denotes differentiation with respect to time.) For isentropic fields, the pressure P is a function only of the density, and the right side of (2) may also be written as $-\boldsymbol{\nabla} V'(\rho)$, where $V'(\rho)$ is the enthalpy, $P(\rho) = \rho V'(\rho) - V(\rho)$, and $\sqrt{\rho V''(\rho)} = \sqrt{P'(\rho)}$ is the sound speed (prime denotes differentiation with respect to argument).

Equations (1) and (2) can be obtained by bracketing the dynamical variables ρ and $\boldsymbol{v}$ with the Hamiltonian $H(\rho, \boldsymbol{v})$

$$H(\rho, \boldsymbol{v}) = \int \mathrm{d}r \big(\tfrac{1}{2}\rho v^2 + V(\rho)\big) \tag{3}$$

$$\dot{\rho} = \{H, \rho\} \tag{4a}$$

$$\dot{\boldsymbol{v}} = \{H, \boldsymbol{v}\} \tag{4b}$$

provided the nonvanishing brackets of the fundamental variables $(\rho, \boldsymbol{v})$ are taken to be [2]

$$\{v^i(\boldsymbol{r}), \rho(\boldsymbol{r}')\} = \partial_i \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{5a}$$

$$\{v^i(\boldsymbol{r}), v^j(\boldsymbol{r}')\} = -\frac{\omega_{ij}(\boldsymbol{r})}{\rho(\boldsymbol{r})}\delta(\boldsymbol{r} - \boldsymbol{r}') \ . \tag{5b}$$

(The fields in the brackets are at equal times, hence the time argument is suppressed.) Here ω_{ij} is the vorticity, defined as the curl of v^i:

$$\omega_{ij} = \partial_i v^j - \partial_j v^i \ . \tag{6}$$

One naturally asks whether there is a canonical 1-form that leads to the symplectic structure (5); that is, one seeks a Lagrangian whose canonical variables can be used to derive (5) from canonical brackets. When the velocity is irrotational, the vorticity vanishes, $\boldsymbol{v}$ can be written as the gradient of a velocity potential θ, $\boldsymbol{v} = \boldsymbol{\nabla}\theta$, and (5) is satisfied by postulating that

$$\{\theta(\boldsymbol{r}), \rho(\boldsymbol{r}')\} = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{7}$$

that is, the velocity potential is conjugate to the density, so that the Lagrangian can be taken as

$$L\big|_{\text{irrotational}} = \int \mathrm{d}r \ \theta\dot{\rho} - H\big|_{\boldsymbol{v}=\boldsymbol{\nabla}\theta} \tag{8}$$

where H is given by (3) with $\boldsymbol{v} = \boldsymbol{\nabla}\theta$.

2 Extending the Formalism to Include Vorticity

The traditional method of including vorticity in a Lagrangian formalism [3] involves writing the velocity in a more elaborate potential representation, the so-called Clebsch parameterization [4],

$$\boldsymbol{v} = \boldsymbol{\nabla}\theta + \alpha\boldsymbol{\nabla}\beta \tag{9}$$

which supports nonvanishing vorticity

$$\omega_{ij} = \partial_i\alpha\,\partial_j\beta - \partial_j\alpha\,\partial_i\beta \ . \tag{10}$$

The Lagrangian

$$L = -\int \mathrm{d}r\,\rho(\dot\theta + \alpha\dot\beta) - H\big|_{\boldsymbol{v}=\boldsymbol{\nabla}\theta+\alpha\boldsymbol{\nabla}\beta} \tag{11}$$

identifies canonical pairs to be $\{\theta,\rho\}$ (as in the irrotational case) and also $\{\beta,\alpha\rho\}$. It then follows that the algebra (5) is satisfied, provided $\boldsymbol{v}$ is given by (9). The quantities (α,β) are called the "Gauss potentials".

The situation here is similar to the electromagnetic force law: The Lorentz equation can be presented in terms of the electric and magnetic field strengths, but a Lagrangian for the motion requires describing the fields in terms of potentials.

3 Some Further Observations on the Clebsch Decomposition of the Vector Field $\boldsymbol{v}$

In three dimensions, (9) involves the same number of functions on the left and right sides of the equality: three. The total number of dynamical variables $(\rho,\boldsymbol{v})$ is even – four – so an appropriate phase space can be constructed from the four potentials $(\rho,\theta,\alpha,\beta)$. Nevertheless the Gauss potentials are not uniquely determined by $\boldsymbol{v}$. The following is the reason why a canonical formulation of (5) requires using the Clebsch decomposition (9). Although the algebra (5) is consistent in that the Jacobi identity is satisfied, it is degenerate in that the kinematic helicity h

$$h \equiv \tfrac{1}{2}\int \mathrm{d}^3r\,\boldsymbol{v}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) = \tfrac{1}{2}\int \mathrm{d}^3r\,\boldsymbol{v}\cdot\boldsymbol{\omega} \tag{12}$$

$(\omega^i = \frac{1}{2}\varepsilon^{ijk}\omega_{jk})$ has vanishing bracket with ρ and $\boldsymbol{v}$. (Note that h is just the Abelian Chern-Simons term of $\boldsymbol{v}$ [5].) Consequently, a canonical formulation requires eliminating the kernel of the algebra, that is, neutralizing h. This is achieved by the Clebsch decomposition: $\boldsymbol{v} = \boldsymbol{\nabla}\theta + \alpha\boldsymbol{\nabla}\beta$, $\boldsymbol{\omega} = \boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\beta$, $\boldsymbol{v}\cdot\boldsymbol{\omega} = \boldsymbol{\nabla}\theta\cdot(\boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\beta) = \boldsymbol{\nabla}\cdot(\theta\boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\beta)$. Thus in the Clebsch parameterization the helicity is given by a surface integral $h = \frac{1}{2}\int \mathrm{d}\boldsymbol{S}\cdot\theta(\boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\beta)$ – it possesses no bulk contribution, and the obstruction to a canonical realization of (5) is removed [6].

In two spatial dimensions, the Clebsch parameterization is redundant, involving three functions to express the two velocity components. Moreover, the kernel of (5) in two dimensions comprises an infinite number of quantities

$$k_n = \int \mathrm{d}^2r\,\rho\Big(\frac{\omega}{\rho}\Big)^n \tag{13}$$

for which the Clebsch parameterization offers no simplification. (Here ω is the two-dimensional vorticity $\omega_{ij} = \varepsilon_{ij}\omega$.) Nevertheless, a canonical formulation in two dimensions also uses Clebsch variables to obtain an even-dimensional phase space.

4 Kinematical Grassmann Variables for Vorticity

Rather than using the Gauss potentials (α, β) of the Clebsch parameterization (9) in the description of vorticity (10), we propose an alternative that makes use of Grassmann variables [7]. We write

$$\boldsymbol{v} = \boldsymbol{\nabla}\theta - \tfrac{1}{2}\psi_a \boldsymbol{\nabla}\psi_a \tag{14}$$

where ψ_a is a multicomponent, real Grassmann spinor $\psi_a^* = \psi_a$, $(\psi_a\psi_b)^* = \psi_a^*\psi_b^*$. (The number of components depends on spatial dimensionality.) Evidently the nonvanishing vorticity is

$$\omega_{ij} = -\partial_i\psi_a\,\partial_j\psi_a\ . \tag{15}$$

Moreover, the canonical 1-form in the Lagrangian that replaces (11) reads

$$L = -\int \mathrm{d}r\,\rho(\dot\theta - \tfrac{1}{2}\psi_a\dot\psi_a) - H\big|_{\boldsymbol{v}=\boldsymbol{\nabla}\theta - \frac{1}{2}\psi\boldsymbol{\nabla}\psi} \tag{16}$$

The Hamiltonian retains its (bosonic) form (3), but the Grassmann variables are hidden in the formula for the velocity. From the canonical 1-form, we deduce that (θ, ρ) remain a conjugate pair [see (7)] and that the canonically independent Grassmann variables are $\sqrt{\rho}\,\psi$. Thus we postulate, in addition to the Poisson bracket (7) satisfied by (θ, ρ), a Poisson antibracket for the Grassmann variables

$$\{\psi_a(\boldsymbol{r}), \psi_b(\boldsymbol{r}')\} = -\frac{\delta_{ab}}{\rho(\boldsymbol{r})}\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{17}$$

and this, together with (7), has the further consequence that the following brackets hold:

$$\{\theta(\boldsymbol{r}), \psi(\boldsymbol{r}')\} = -\frac{1}{2\rho(\boldsymbol{r})}\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{18}$$

$$\{\boldsymbol{v}(\boldsymbol{r}), \psi(\boldsymbol{r}')\} = -\frac{\boldsymbol{\nabla}\psi(\boldsymbol{r})}{\rho(\boldsymbol{r})}\delta(\boldsymbol{r} - \boldsymbol{r}')\ . \tag{19}$$

The algebra (5) follows.

One may state that it is natural to describe vorticity by Grassmann variables: vortex motion is associated with spin, and the Grassmann description of spin within classical physics is well known. In the model as developed thus far the Grassmann variables have no role beyond the kinematical one of parameterizing vorticity (15) and providing the correct bracket structure. They do not contribute to the equations of motion for ρ and $\boldsymbol{v}$, (1) and (2) [even though they are hidden in the formula (14) for $\boldsymbol{v}$]. Moreover, they satisfy a free equation: from (16) it follows that

$$\dot\psi + \boldsymbol{v}\cdot\boldsymbol{\nabla}\psi = 0\ . \tag{20}$$

5 Dynamical Grassmann Variables for Supersymmetry

Thus far the Grassmann variables' only role has been to parameterize the velocity/vorticity (14), (15) and to provide canonical variables for the symplectic structure (5). The equations for the fluid (1), (2) are not polluted by them and they do not appear in the Hamiltonian, beyond their hidden contribution to $\boldsymbol{v}$. Thus the equation for the Grassmann fields is free (20).

But now we enquire whether we can add a Grassmann term to the Hamiltonian so that the Grassmann variables enter the dynamics and the entire model enjoys supersymmetry.

We have succeeded for a specific form of the potential $V(\rho)$:

$$V(\rho) = \frac{\lambda}{\rho} \tag{21}$$

and for the specific dimensionalities of space-time: (2+1) and (1+1). The reason for these specificities will be explained in the next Section.

The potential (21), with $\lambda > 0$, leads to negative pressure

$$P(\rho) = \rho V'(\rho) - V(\rho) = -2\lambda/\rho \tag{22}$$

and sound speed

$$s = \sqrt{P'(\rho)} = \sqrt{2\lambda}/\rho \tag{23}$$

(hence $\lambda > 0$). This model is called the "Chaplygin gas".

Chaplygin introduced his equation of state as a mathematical approximation to the physically relevant adiabatic expressions $V(\rho) \propto \rho^n$ with $n > 0$ [8]. (Constants are arranged so that the Chaplygin formula is tangent at one point to the adiabatic profile.) Also it was realized that certain deformable solids can be described by the Chaplygin equation of state [9]. These days negative pressure is recognized as a possible physical effect: exchange forces in atoms give rise to negative pressure; stripe states in the quantum Hall effect may be a consequence of negative pressure; the recently discovered cosmological constant may be exerting negative pressure on the cosmos, thereby accelerating expansion.

5.1 Planar model

In (2+1) dimensions the Grassmann variables possess 2-components and two real 2×2 Dirac "α"-matrices act on them: $\alpha^1 = \sigma^1$, $\alpha^2 = \sigma^3$. The supersymmetric Hamiltonian is

$$H = \int \mathrm{d}^2 r \Big\{ \tfrac{1}{2}\rho v^2 + \frac{\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2} \psi \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \psi \Big\} \tag{24}$$

where it is understood that $\boldsymbol{v} = \boldsymbol{\nabla}\theta - \frac{1}{2}\psi \boldsymbol{\nabla}\psi$ [7, 10]. While the continuity equation retains its form (1), the force equation acquires a contribution from the Grassmann variables

$$\dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = \boldsymbol{\nabla} \frac{\lambda}{\rho^2} + \frac{\sqrt{2\lambda}}{\rho} (\boldsymbol{\nabla}\psi) \boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\psi \tag{25}$$

and ψ is no longer free:

$$\dot{\psi} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \psi = \frac{\sqrt{2\lambda}}{\rho} \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \psi \ . \tag{26}$$

These equations of motion, together with (1), ensure that the following supercharges are time independent:

$$Q = \int \mathrm{d}^2 r \{\rho \boldsymbol{v} \cdot (\boldsymbol{\alpha} \psi) + \sqrt{2\lambda} \psi\} \tag{27a}$$

$$\tilde{Q} = \int \mathrm{d}^2 r \, \rho \psi \ . \tag{27b}$$

They generate the following transformations:

$$\delta \rho = -\boldsymbol{\nabla} \cdot \big[\rho(\eta \boldsymbol{\alpha} \psi)\big] \qquad \tilde{\delta} \rho = 0 \tag{28a}$$

$$\delta \psi = -(\eta \boldsymbol{\alpha} \psi) \cdot \boldsymbol{\nabla} \psi - \boldsymbol{v} \cdot \boldsymbol{\alpha} \eta - \frac{\sqrt{2\lambda}}{\rho} \eta \qquad \tilde{\delta} \psi = -\eta \tag{28b}$$

$$\delta \boldsymbol{v} = -(\eta \boldsymbol{\alpha} \psi) \cdot \boldsymbol{\nabla} v + \frac{\sqrt{2\lambda}}{\rho} (\eta \boldsymbol{\nabla} \psi) \qquad \tilde{\delta} \boldsymbol{v} = 0 \tag{28c}$$

where η is a two-component constant Grassmann spinor. The antibrackets of the supercharges produce other conserved quantities:

$$\{Q_a, Q_b\} = -2\delta_{ab} H \tag{29a}$$

$$\{\tilde{Q}_a, \tilde{Q}_b\} = -\delta_{ab} N \tag{29b}$$

$$\{\tilde{Q}_a, Q_b\} = \boldsymbol{\alpha}_{ab} \cdot \boldsymbol{P} + \sqrt{2\lambda} \delta_{ab} \Omega \ . \tag{29c}$$

Here N is the conserved number $\int \mathrm{d}^2 r \, \rho$, $\boldsymbol{P}$ is the conserved momentum $\int \mathrm{d}^2 r \, \rho \boldsymbol{v}$, and Ω is a center given by the volume of space $\int \mathrm{d}^2 r$.

5.2 Lineal model

In (1+1) dimensions the Chaplygin gas equation can be written in compact form in terms of the Riemann coordinates

$$R_{\pm} = v \pm \sqrt{2\lambda}/\rho \ . \tag{30}$$

Both eqs. (1) and (2) are equivalent to

$$\dot{R}_{\pm} = -R_{\mp} \frac{\partial}{\partial x} R_{\pm} \ . \tag{31}$$

It is known that this system is completely integrable [11]. One hint for this is the existence of an infinite number of constants of motion:

$$I_{\pm}^{n} = \int \mathrm{d}x \, \rho (R_{\pm})^{n} \tag{32}$$

are time-independent by virtue of (31).

The supersymmetric Hamiltonian makes use of a real, 1-component Grassmann field ψ [12]:

$$H = \int \mathrm{d}x \Big(\tfrac{1}{2}\rho v^2 + \frac{\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2} \psi \frac{\partial}{\partial x} \psi \Big) \ . \tag{33}$$

The velocity is given by $v = \frac{\partial}{\partial x}\theta - \frac{1}{2}\psi\frac{\partial}{\partial x}\psi$ and the equations of motion for the bosonic variables retain the same form as the absence of ψ, that is, (1), (2) continue to hold. The Grassmann field satisfies

$$\dot{\psi} + R_- \frac{\partial}{\partial x} \psi = 0 \tag{34}$$

and a general solution follows immediately with the help of (31): ψ is an arbitrary function of R_+.

$$\psi = \Psi(R_+) \tag{35}$$

Thus the system remains completely integrable.

The supersymmetry charges and transformation laws are obvious dimensional reductions of (27)–(28):

$$Q = \int \mathrm{d}x \, \rho R_+ \psi \tag{36a}$$

$$\tilde{Q} = \int \mathrm{d}x \, \rho \psi \tag{36b}$$

$$\delta\rho = -\eta \frac{\partial}{\partial x}(\rho\psi) \qquad \tilde{\delta}\rho = 0 \tag{37a}$$

$$\delta\psi = -\eta\psi\psi' - \eta R_+ \qquad \tilde{\delta}\psi = -\eta \tag{37b}$$

$$\delta v = -\eta(\psi v)' + \eta R_+ \psi' \qquad \tilde{\delta} v = 0 \ . \tag{37c}$$

The algebra of these is

$$\{Q, Q\} = -2H \tag{38a}$$

$$\{\tilde{Q}, \tilde{Q}\} = -N \tag{38b}$$

$$\{\tilde{Q}, Q\} = P + \sqrt{2\lambda}\Omega \ . \tag{38c}$$

In view of (35), we see that evaluating the supercharges Q and $\tilde{Q}$ on the solution gives expressions of the same form as the bosonic conserved charges (33).

Indeed, we recognize that two charges in (36) are the first two in an infinite tower of conserved supercharges, which generalizes the infinite number of bosonic conserved quantities (32):

$$Q_n = \int \mathrm{d}x \, \rho R_+^n \psi \ . \tag{39}$$

6 The Origins of Our Models

We have succeeded in supersymmetrizing a specific model – the Chaplygin gas – in specific dimensionalities – the 2-dimensional plane and the 1-dimensional line – leading to nonrelativistic, supersymmetric fluid mechanics in (2+1)- and (1+1)-dimensional space-time. The reason for these specificities is that both models descend from Nambu-Goto models for extended systems in a target space of one dimension higher than the world volume of the extended object. Specifically, a membrane in three spatial dimensions and a string in two spatial dimensions, when gauge-fixed in a light-cone gauge, can be shown to devolve to a bosonic Chaplygin gas in two and one spatial dimensions, respectively [13]. The fluid velocity potential arises from the single dynamical variable in the gauge-fixed Nambu-Goto theory, namely, the transverse direction variable for the membrane in space and the string on a plane. Although purely bosonic Chaplygin gas models in other dimensions can devolve from appropriate Nambu-Goto models for extended objects, for the supersymmetric case we need a superextended object, and these exist only in specific dimensionalities. In our case it is the light-cone parameterized supermembrane in (3+1)-dimensional space-time [14] and the superstring in (2+1)-dimensional space-time [15] that give rise to our planar and lineal supersymmetric fluid models.

One naturally wonders whether an arbitrary bosonic potential $V(\rho)$ has a supersymmetric partner in arbitrary dimensions, and this problem is under further investigation. One promising approach is to consider parameterizations of extended objects other than the light-cone one. It is known that in the purely bosonic case, other parameterizations of the Nambu-Goto actions lead to other fluid mechanical models, and this should carry over to a supersymmetric generalization.

Incidentally, the existence of Nambu-Goto antecedents of the fluid models that we have discussed allows one to understand some of their remarkable properties: complete integrability in the lineal case; existence of further symmetries (which we have not discussed here) and relation to other models (which devolve from the same extended system, but are parameterized differently from the light-cone method) [16].

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